Robustly Minimax Codes for Universal Data Compression

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Abstract— We introduce a notion of 'relative redundancy' for universal data compression and propose a universal code which asymptotically achieves the minimax value of the relative redundancy. The relative redundancy is a hybrid of redundancy and coding regret (pointwise redundancy), where a class of information sources and a class of codes are assumed. The minimax code for relative redundancy is an extension of the modified Jeffreys mixture, which was introduced by Takeuchi and Barron and is minimax for regret.

Keywords— universal coding, redundancy, regret, Bayes mixture, Jeffreys prior, robust learning

1 Introduction

We introduce a notion of 'relative redundancy' for universal data compression and sequential prediction, and propose procedures which asymptotically achieve the minimax value of the relative redundancy.

For universal data compression, the notion of redundancy, which is the difference between the expected code length of the code in concern and the expected code length of the Shannon code defined by the true source, has been often used as a criterion. On the other hand these days, there is a different criterion called 'regret' or 'pointwise redundancy', for which any true information source is not assumed [10]. Instead, a class of codes, which is the class of competitors for the code in concern, is assumed. Given a data sequence, the regret of a code is defined as the difference between the code length by the code in concern and the code length by the hindsight best code for the data sequence among the assumed class of codes. The relative redundancy is a hybrid of those two notions of redundancy.

For the relative redundancy, we assume both a class of sources and a class of codes. (In the case of ordinary redundancy, the both are same to each other.) The relative redundancy is the difference between the expected code length of the code in concern and the shortest expected code length achieved by a code among the assumed class of codes, where expectation is taken with respect to the true source, an element of the class of sources. We are especially interested in the case that the class of sources is non parametric and properly includes the class of codes. This setting requires the codes to be robust.

When extending the notion of redundancy as the above, the minimax codes for traditional redundancy obtained by Bayes procedure with Jeffreys prior [5, 6, 13] are no longer minimax. Similarly, the minimax codes for regret obtained by normalizing maximum likelihood [10, 9] or Bayes procedures [14, 11, 4, 12] are not minimax. Takeuchi and Barron [12] showed that the minimax codes for redundancy are not minimax for the regret in usual cases and they obtained a minimax code, which are a mixture of enlarged class of codes. This enlargement is obtained by using the difference between the Fisher information and the empirical Fisher information. Usually, the above code is not minimax for our relative redundancy and we need another modification. The minimax code for the relative redundancy we propose is a mixture over an enlarged class of codes, which is obtained by utilizing the difference between the empirical covariance of score functions and the Fisher information, adding the difference between the Fisher information and the empirical Fisher information. It might be interesting that the value of the asymptotic minimax relative redundancy is same as that of the asymptotic minimax redundancy upto constant order.

In the field of computational learning theory, this kind of robust setting was proposed for the batch learning scenario and is known as 'agnostic PAC (Probably Approximate Correct) learning model' [8]. Also, similar setting is used for the problem of sequential prediction [15], where general loss functions rather than code length are used. Our problem with the relative redundancy can be thought of as a special case of that, though any minimax procedures upto constant order were not known to date.

2 Relative Redundancy

Let $\mathcal{X}$ be a measurable space and ν be a reference measure on $\mathcal{X}$. We define $\nu(dx^n) \stackrel{\rm def}{=} \prod_{t=1}^n \nu(dx_t)$. We refer to p as the density of a stochastic process, if p satisfies $\int p(x_1)\nu(dx) = 1$ and $\int p(x^{n+1})\nu(dx_{n+1}) = p(x^n)$.

For now we assume that $\mathcal{X}$ is discreet and $\nu(\{x\}) = 1$ for $x \in \mathcal{X}$. Let q be a density of stochastic process. We can construct a code for the set of data $\mathcal{X}^n$ whose code length is given by $-\log q(x^n)$. Here, log is the natural logarithm. We measure code length by 'nat'. Conversely, when there exists a uniquely decodable code for $\mathcal{X}^n$ with code length $l(x^n)$, then $\int \exp(-l(x^n))\nu(dx^n) \leq 1$ (Kraft's inequality) holds. Hence, we refer to the density q as a code. We let C be a class of codes. We assume that C is a smooth parametric class: $C \stackrel{\rm def}{=} \{p(\cdot|\theta) : \theta \in \Theta \subset \Re^d\}$. The circumstances for general $\mathcal{X}$ and ν are similar.

Let S be a certain class of densities of stochastic

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processes. We assume that data sequence $x^n \in \mathcal{X}^n$ is drawn from a certain $p \in S$, i.e. we refer to S as the class of information sources.

Let r be the true source, i.e. let r be an element of S. Let q be a code. Define the relative redundancy of the code q with respect to the true source r and the class of codes C as

$$R_n(q, r, C) \stackrel{\text{def}}{=} E_r \log \frac{1}{q(x^n)} - \inf_{p \in C} E_r \log \frac{1}{p(x^n)}.$$

Define the worst case relative redundancy of q for the pair (S, C) as

$$R_n(q, S, C) \stackrel{\text{def}}{=} \sup_{r \in S} \Big(E_r \log \frac{1}{q(x^n)} - \inf_{p \in C} E_r \log \frac{1}{p(x^n)} \Big).$$

When S equals C, the relative redundancy coincides with the ordinary redundancy. Finally, the minimax relative redundancy for the pair (S, C) is defined as

$$\bar{R}_n(S, C) \stackrel{\text{def}}{=} \inf_q \sup_{r \in S} \Big(E_r \log \frac{1}{q(x^n)} - \inf_{p \in C} E_r \log \frac{1}{p(x^n)} \Big).$$

The above definition is valid for general classes of stochastic processes, but hereafter, we restrict C and S to classes of i.i.d. processes, i.e. we assume that $p(x^n) = \prod_{t=1}^n p(x_t)$ for $p \in C \cup S$.

In that case, it is known that $\bar{R}_n(C, C) = (d/2) \log(n/2\pi e) + \log C_J(\Theta) + o(1)$ holds [5, 6, 13]. Here, $o(1) \to 0$ and $C_J(\Theta) \stackrel{\text{def}}{=} \int_\Theta \sqrt{\det J(\theta)} d\theta$, where $J(\theta)$ is the Fisher information matrix defined as

$$J_{ij}(\theta) \stackrel{\text{def}}{=} E_\theta(-\frac{\partial^2 \log p(x|\theta)}{\partial \theta^i \partial \theta^j}).$$

In this paper, we show that $\bar{R}_n(S, C) = (d/2) \log(n/2\pi e) + \log C_J(\Theta) + o(1)$ still holds for fairly general i.i.d. classes of codes C.

We describe the definition of minimax regret $\bar{r}$ for reference, where W_n is a subset of $\mathcal{X}^n$:

$$\bar{r}_n(W_n, C) \stackrel{\text{def}}{=} \inf_q \sup_{x^n \in W_n} \Big(\log \frac{1}{q(x^n)} - \inf_{p \in C} \log \frac{1}{p(x^n)} \Big).$$

For this, it is known [9, 14, 11, 4, 12, 3] that $\bar{r}_n(W_n, C) = (d/2) \log(n/2\pi) + \log C_J(\Theta) + o(1)$, where $W_n = \{x^n : \hat{\theta}(x^n) \in \Theta\}$ and $\hat{\theta}(x^n)$ is the maximum likelihood estimate given x^n.

3 Minimax Code

Let $\mathcal{P}$ be a set of all i.i.d. processes and C be a smooth parametric subset of $\mathcal{P}$: $C = \{p(\cdot|\theta) : \theta \in \Theta_c \subset \Theta\}$, where Θ_c is compact and included in Θ°. We refer to C as a class of codes. We define $\tilde{\theta}$ for $r \in \mathcal{P}$ as

$$\tilde{\theta} = \tilde{\theta}_r \stackrel{\text{def}}{=} \arg\min_{\theta \in \Theta} E_r \log \frac{1}{p(x|\theta)}.$$

Note that we have $E_r \nabla \log p(x|\tilde{\theta}) = 0$, where we let ∇ denote the gradient with respect to θ. Define a set of probability densities as $S \stackrel{\text{def}}{=} \{r : \tilde{\theta}_r \in \Theta_c\}$. We will consider the relative redundancy for pair (S, C) (actually we must consider the problem for (S', C) where S' is a certain subset of S).

Let K_n be a compact subset of Θ such that $K_n^\circ \supset \Theta_c$ $(n \geq 0)$, $K_n \subset K_0^\circ$ $(n \geq 1)$ and K_n slowly shrinks to Θ_c as $n \to \infty$.

Now, we describe the conditions under which we construct the minimax code. Define a matrix $\hat{I}$ as

$$\hat{I}(x, \theta) \stackrel{\text{def}}{=} \frac{\partial \log p(x|\theta)}{\partial \theta^i} \frac{\partial \log p(x|\theta)}{\partial \theta^j}$$

and $\hat{I}(x^n, \theta) \stackrel{\text{def}}{=} (1/n) \sum_{t=1}^n \hat{I}(x_t, \theta)$. We assume $E_\theta \hat{I}(x, \theta) = J(\theta)$. Define a d^2-dimensional vector valued random variable $V(x, \theta)$ as $V_{dj+i}(x^n, \theta) \stackrel{\text{def}}{=} \hat{I}_{ij}(x^n, \theta) - J_{ij}(\theta)$. Note that $E_\theta V(x, \theta) = 0$. Define $I(r, \theta) \stackrel{\text{def}}{=} E_r \hat{I}(x, \theta)$. In particular, we let $I(r) = I(r, \tilde{\theta}_r)$. Note that $I(p(\cdot|\theta)) = J(\theta)$.

Define a d^2-dimensional vector valued random variable

$$U_{dj+i}(x^n, \theta) \stackrel{\text{def}}{=} \hat{J}_{ij}(x^n, \theta) - J_{ij}(\theta),$$

where $\hat{J}(x^n, \theta)$ is the empirical Fisher information:

$$\hat{J}_{ij}(x^n, \theta) \stackrel{\text{def}}{=} -\frac{1}{n} \sum_{t=1}^n \frac{\partial^2 \log p(x_t|\theta)}{\partial \theta^i \partial \theta^j}.$$

Condition 1 *Let $D = [-b, b]^{d^2}$ for a certain $b > 0$. For a certain $C_0 > 0$, the following holds.*

$$\begin{aligned} &\forall (u, v) \in D^2, \ \forall \theta \in K_0, \\ &E_\theta \exp(v \cdot V(x, \theta) + u \cdot U(x, \theta)) < C_0. \end{aligned} \tag{1}$$

Note that the above implies the following for any $\gamma > 0$.

$$E_\theta(|V(x, \theta)|^\gamma + |U(x, \theta)|^\gamma) e^{v \cdot V(x,\theta) + u \cdot U(x,\theta)} < C(\gamma).$$

We consider the class of sources S' which satisfies the following.

Condition 2 *For a certain C_0, the following holds.*

$$\forall r \in S', \ E_r(|U(x, \theta)|^2 + |V(x, \theta)|^2) < C_0. \tag{2}$$

Under Condition 1, we define an extended class $\bar{C}$:

$$\bar{C} \stackrel{\text{def}}{=} \{p_e(\cdot|\theta, u, v) = \frac{p(\cdot|\theta) e^{u \cdot U(\cdot,\theta) + v \cdot V(\cdot,\theta)}}{\lambda(\theta, u, v)} : \theta \in K_0, \ (u, v) \in D^2\},$$

where $\lambda(\theta, u, v) \stackrel{\text{def}}{=} E_\theta e^{u \cdot U(x,\theta) + v \cdot V(x,\theta)}$. This is the normalization constant. Note that $\bar{C}$ is $(d + 2d^2)$-dimensional and the original class C is smoothly embedded in the enlarged class $\bar{C}$.

Let m_{K_t} be Bayes mixture $\int_{K_t} p(\cdot|\theta) w_{K_t}(\theta) d\theta$ with Jeffreys prior: $w_{K_t}(\theta) \stackrel{\text{def}}{=} \sqrt{\det J(\theta)} / C_J(K_t)$.

Let $\xi \stackrel{\text{def}}{=} (\theta, u, v)$. The following is our new code.

$$m_n(x^n) \stackrel{\text{def}}{=} (1 - \epsilon_n) m_{K_n}(x^n) + \epsilon_n \int p_e(x^n|\xi) w(\xi) d\xi,$$

where $w(\xi)$ is some smooth prior with $\inf_\xi w(\xi) > 0$ and $\epsilon_n > 0$ approaches to 0 at polynomial rate.

Remark: If $\mathcal{X}$ is a finite set, we can use the class $\mathcal{P}$ as the enlarged class $\bar{C}$, which is finite dimensional.

We have the following theorem, which claims that the above m_n is asymptotically minimax for relative redundancy.

Theorem 1 *Under Conditions 1 and 2, and certain regularity conditions for C, the following holds.*

$$\limsup_{n\to\infty}(R_n(m_n, S', C) - \frac{d}{2}\log\frac{n}{2\pi e}) \le \log C_J(\Theta_c).$$

We note about the lower bound. Whenever $S' \supseteq C$, $R_n(q, C, C) \le R_n(q, S', C)$ holds. Hence, a lower bound for the minimax relative redundancy is that for the minimax relative redundancy and we can see that the above upper bound matches the lower bound.

Outline of the Proof

We make a case argument about r. Let $a_n = n^{-1/4}$. There are two cases: (i) $|E_r U(x, \tilde{\theta}_r)| + |E_r V(x, \tilde{\theta}_r)| \le 2a_n$ holds. (ii) $|E_r U(x, \tilde{\theta}_r)| \ge a_n$ or $|E_r V(x, \tilde{\theta}_r)| \ge a_n$ hold.

We let $S_n^{(i)}$ and $S_n^{(ii)}$ respectively denote the sets of $r \in S'$ which satisfies the cases (i) and (ii). We use the notation $l_n = (1/n)\nabla \log p(x^n|\tilde{\theta}_r)$.

First, consider the case (i). In this case, we follow the argument used in [5, 6]. We have $|\hat{J}(x^n, \tilde{\theta}) - J(\tilde{\theta})| = |U(x^n, \tilde{\theta})| \le 4a_n = o(1)$ with high probability. Hence the Laplace integration about m_{K_n} works. The following holds with high probability, where we let $\delta\theta = \theta - \tilde{\theta}$.

$$\begin{aligned}\frac{1}{n}\log\frac{p(x^n|\theta)}{p(x^n|\tilde{\theta})} &= \delta\theta^t l_n - \frac{\delta\theta^t \hat{J}(x^n, \theta')\delta\theta}{2} + O(|\delta\theta|^3) \\ &= \delta\theta^t l_n - \frac{\delta\theta^t J(\tilde{\theta})\delta\theta}{2} + O(|\delta\theta^t|^3),\end{aligned}$$

where $\theta' = \lambda\theta + (1-\lambda)\tilde{\theta}$ for a certain $\lambda \in [0, 1]$. Let $h \stackrel{\text{def}}{=} (J(\tilde{\theta}))^{-1} l_n$ and $\tilde{J}$ denote $J(\tilde{\theta})$, then we have

$$\begin{aligned}&\frac{1}{n}\log\frac{p(x^n|\theta)}{p(x^n|\tilde{\theta})} \\ =\ & -\frac{(\delta\theta - h)^t \tilde{J}(\delta\theta - h)}{2} + \frac{h^t \tilde{J} h}{2} + O(|\delta\theta|^3) \\ =\ & -\frac{(\delta\theta^t - h)^t \tilde{J}(\delta\theta^t - h)}{2} + \frac{l_n^t \tilde{J}^{-1} l_n}{2} + O(|\delta\theta|^3).\end{aligned}$$

Hence we have

$$\frac{p(x^n|\theta)}{p(x^n|\tilde{\theta})} = e^{-n((\theta-\tilde{\theta}-h)^t \tilde{J}(\theta-\tilde{\theta}-h) - l_n^t \tilde{J}^{-1} l_n + O(|\delta\theta|^3))/2}.$$

Note that $|h| \le \log n/n$ with high probability. Evaluating the integration $\int p(x^n|\theta) w_{K_n}(\theta) d\theta / p(x^n|\tilde{\theta})$ with contribution from the neighborhood of $\tilde{\theta} + h$, we have

$$\frac{m_{K_n}(x^n)}{p(x^n|\tilde{\theta})} \sim \frac{e^{n l_n^t \tilde{J}^{-1} l_n/2}(2\pi)^{d/2}}{C_J(K_n) n^{d/2}}$$

with high probability. Hence, we have

$$\frac{m_n(x^n)}{p(x^n|\tilde{\theta})} \gtrsim \frac{(1-\epsilon_n) e^{n l_n^t \tilde{J}^{-1} l_n/2}(2\pi)^{d/2}}{C_J(K_n) n^{d/2}}.$$

Hence, the following holds with high probability.

$$\log\frac{p(x^n|\tilde{\theta})}{m_n(x^n)} \lesssim \frac{d}{2}\log\frac{n}{2\pi} + \log C_J(\Theta_c) - \frac{n l_n^t \tilde{J}^{-1} l_n}{2}.$$

Noting $E_r l_n l_n^t = I(r)/n$, it is possible to show

$$\begin{aligned}&E_r \log\frac{p(x^n|\tilde{\theta})}{m_n(x^n)} \\ \lesssim\ & \frac{d}{2}\log\frac{n}{2\pi} + \log C_J(\Theta_c) - \frac{\mathrm{tr}(I(r)\tilde{J}^{-1})}{2}.\end{aligned} \tag{3}$$

When $r \in S_n^{(i)}$, we have $|I(r) - J(\tilde{\theta})| = |E_r V(x^n, \tilde{\theta})| \le 2a_n$. Hence,

$$\sup_{r\in S_n^{(i)}} (-\mathrm{tr}(I(r)\tilde{J}^{-1})/2) \sim -d/2. \tag{4}$$

Therefore, we have

$$\sup_{r\in S_n^{(i)}} E_r \log\frac{p(x^n|\tilde{\theta})}{m_n(x^n)} \lesssim \frac{d}{2}\log\frac{n}{2\pi e} + \log C_J(\Theta_c). \tag{5}$$

Now, we consider the case (ii). We have $|U(x^n, \tilde{\theta})| + |V(x^n, \tilde{\theta})| \ge a_n/2$ with high probability. Let

$$(\tilde{u}, \tilde{v}) = \frac{\alpha a_n (U(x^n, \tilde{\theta}), V(x^n, \tilde{\theta}))}{\sqrt{|U(x^n, \tilde{\theta})|^2 + |V(x^n, \tilde{\theta})|^2}},$$

where α is a certain small positive number. Then,

$$\frac{p_e(x^n|\tilde{\theta}, \tilde{u}, \tilde{v})}{p(x^n|\tilde{\theta})} = \frac{e^{n(\tilde{u}\cdot U(x^n, \tilde{\theta}) + \tilde{v}\cdot V(x^n, \tilde{\theta}))}}{(\Lambda(\tilde{\theta}, \tilde{u}, \tilde{v}))^n} \ge e^{C_1 n a_n^2} \tag{6}$$

holds with high probability. We can easily show that

$$\frac{\int p_e(x^n|\xi) w(\xi) d\xi}{p_e(x^n|\tilde{\theta}, \tilde{u}, \tilde{v})} \ge \frac{C_2}{n^{d+2d^2}}.$$

Therefore,

$$\begin{aligned}&\frac{\epsilon_n \int p_e(x^n|\xi) w(\xi) d\xi}{p(x^n|\tilde{\theta})} \\ =\ & \frac{\epsilon_n \int p_e(x^n|\xi) w(\xi) d\xi}{p_e(x^n|\tilde{\theta}, \tilde{u}, \tilde{v})} \frac{p_e(x^n|\tilde{\theta}, \tilde{u}, \tilde{v})}{p(x^n|\tilde{\theta})} \\ \ge\ & \frac{C_2 \epsilon_n \exp(C_1 n a_n^2)}{n^{d+2d^2}} = \frac{C_2 \epsilon_n \exp(C_1\sqrt{n})}{n^{d+2d^2}}\end{aligned}$$

holds with high probability. Hence for $r \in S_n^{(ii)}$,

$$E_r \log\frac{m_n(x^n)}{p(x^n|\tilde{\theta})} \ge E_r \log\frac{\epsilon_n \int p(x^n|\xi) w(\xi) d\xi}{p(x^n|\tilde{\theta})} \to \infty$$

holds. Together with (5), this implies

$$\sup_{r\in S'} E_r \log\frac{p(x^n|\tilde{\theta})}{m_n(x^n)} \le \frac{d}{2}\log\frac{n}{2\pi e} + \log C_J(\Theta_c) + o(1).$$

4 Discussion

4.1 Semi Universality

The minimax codes have the property of 'semi universality' ([7]). Let us take the code based on the Bayes mixture with the Jeffreys prior m_{K_n}, which is asymptotically minimax for the redundancy $R_n(q, C, C)$. Its expected code length per source symbol approaches to the entropy rate of the true source, when the true source r belongs to the class C. This dose not hold, when r is not an element of C. For m_{K_n}, (3) holds as an (approximated) equality rather than an inequality, i.e. we have

$$E_r \log \frac{p(x^n|\tilde{\theta})}{m_{K_n}(x^n)} = \frac{d}{2} \log \frac{n}{2\pi} + O(1).$$

Hence, we have

$$\lim_{n\to\infty} \frac{1}{n} E_r \log \frac{1}{m_{K_n}(x^n)} = \lim_{n\to\infty} \frac{1}{n} E_r \log \frac{1}{p(x^n|\tilde{\theta})}$$

$$= E_r \log \frac{1}{p(x|\tilde{\theta})} = H(r) + D(r|p(\cdot|\tilde{\theta})),$$

where H denotes differential entropy rate and D denotes the Kullback Leibler divergence. Hence, the expected code length by m_{K_n} per source symbol does not converge to the entropy rate. This property is called as 'semi universality'.

Concerning m_n, when $r \notin C$, the relative redundancy is negative. Hence, the code length by m_n is shorter than that by $p(\cdot|\tilde{\theta})$, but the expected code length per source symbol does not converge to the entropy rate.

4.2 Necessity of Enlarging Model

We have succeeded to construct the asymptotically minimax code for the relative redundancy by enlarging the class of codes. Here, we consider why this enlargement is needed. Two information matrices, $\hat{J}(x^n, \tilde{\theta})$ and $I(r)$ are important. When the true source r belongs to the class of codes C, then both $I(r)$ and the expectation of $\hat{J}(x^n, \tilde{\theta})$ equal the Fisher information $J(\tilde{\theta})$. Then, the asymptotics (3) and (4) hold. However, if r is displaced from C, then I and $\hat{J}$ is different from J with high probability. This spoils (3) and (4). However in that case, the contribution from enlarged class works well, utilizing $\hat{I} - J$ or $\hat{J} - J$. When we treat not the relative redundancy but the regret, then we do not have to care the asymptotic (4). Therefore, the enlargement for the regret uses $\hat{J} - J$ alone.

Finally, the authors would like to briefly note about the differential geometrical interpretation (see [1, 2]). The quantity $\hat{J} - J$ relates to exponential curvature of the class C. When $\hat{J}(x^n, \hat{\theta}) - J(\hat{\theta})$ always equals zero, then C is an exponential family. Also, the quantity $\hat{I} - \hat{J}$ relates to the mixture curvature of the class C. When $\hat{I}(x^n, \hat{\theta}) - \hat{J}(x^n, \hat{\theta})$ always equals zero, C is a finite mixture model. Since our enlargement is spanned by $\hat{J} - J$ and $\hat{I} - J$, this is equivalent to the enlargement spanned by $\hat{J} - J$ and $\hat{I} - \hat{J}$. Therefore, our enlargement is to the direction of both exponential and mixture curvatures.

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