Introduction to Polar Codes

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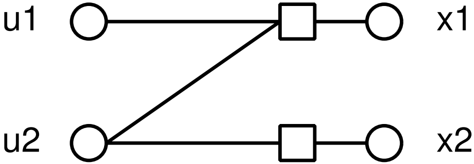
Polar codes [Arıkan 2008]

Capacity achieving codes with efficient encoding and decoding algorithms for any symmetric channels

- $O(N \log N)$ encoding and decoding complexity for blocklength N.
- $o(\mathrm{e}^{-N^{\frac{1}{2}-\epsilon}})$ error probability for any $R < I(W)$ and $\epsilon > 0$.

The 2×2 matrix

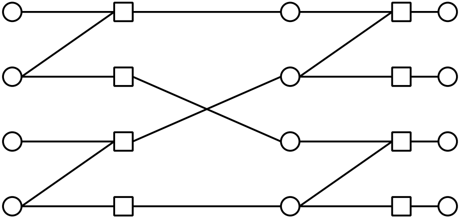
$$G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

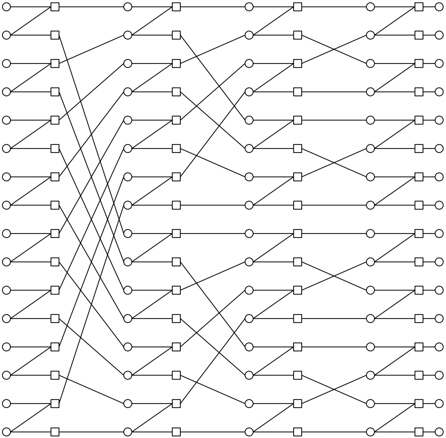
Kronecker product

$$G_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\otimes 2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

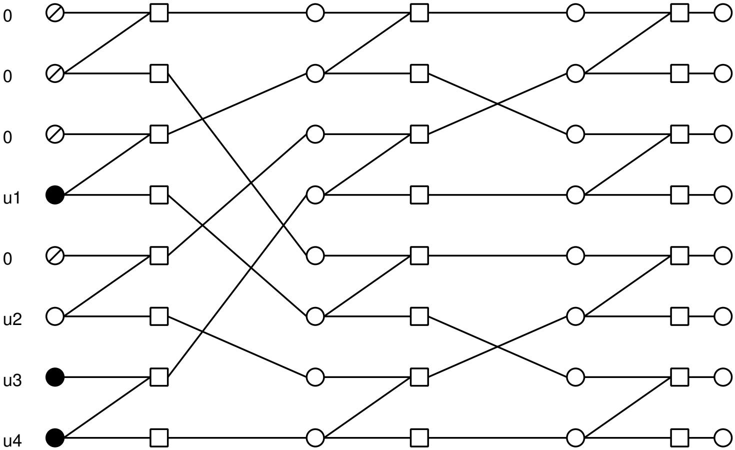


Kronecker product

$$G_{2^n} = (I_{2^{n-1}} \otimes G_2) R_{2^n} (I_2 \otimes G_{2^{n-1}}) = B_{2^n} G_2^{\otimes n}$$

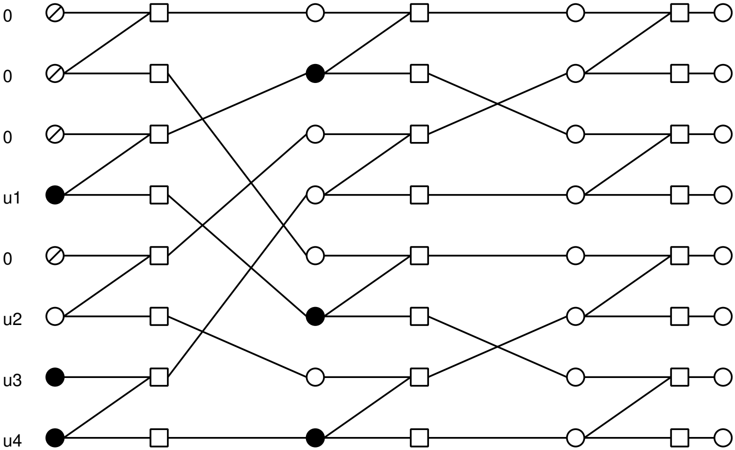


Encoding of polar codes



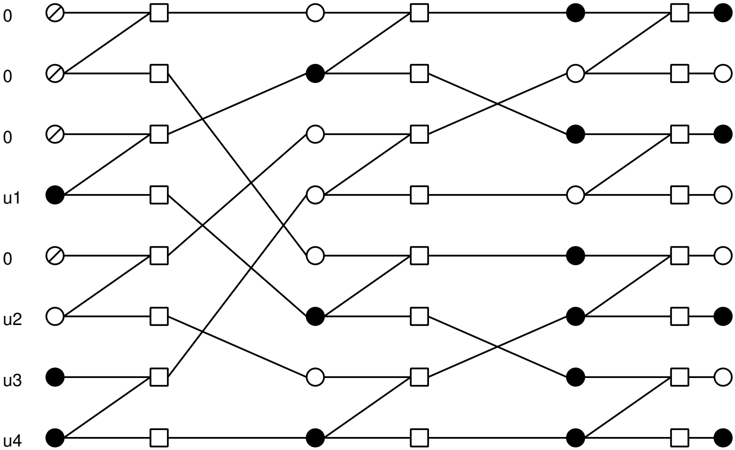
∅: frozen bit # check nodes $= N \log N$

Encoding of polar codes



$\oslash$: frozen bit $\quad \#$ check nodes $= N \log N$

Encoding of polar codes



Encoding complexity $\propto$ # check nodes $= N \log N$

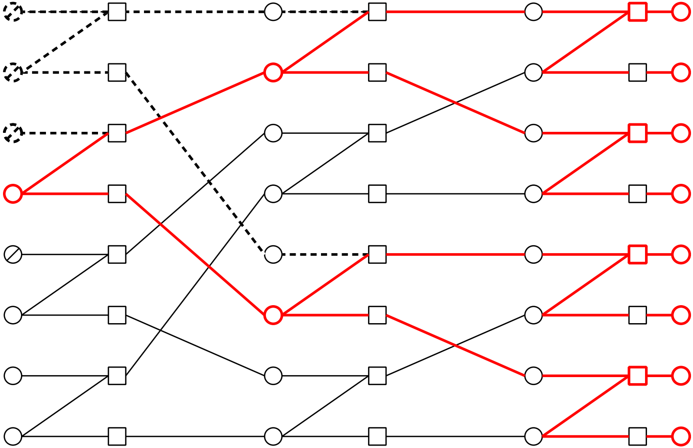
Decoding of polar codes

Successive cancellation (SC) decoding: Sequential decoding from the top to the bottom

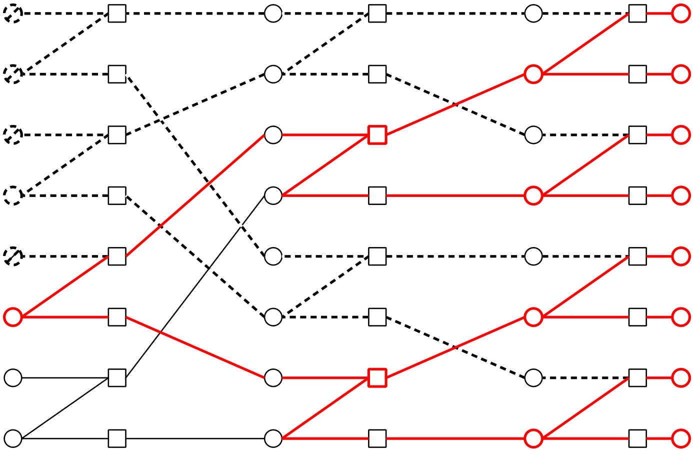
F: The index set of frozen bits

- If $i \in F$, $\hat{u}_i = 0$.
- If $i \notin F$, $\hat{u}_i = \arg\max_{0,1} W^{(i)}(\hat{u}_0^{i-1}, y_0^{N-1} \mid u_i)$.

Decoding of polar codes



Decoding of polar codes



ML on a tree $\Longleftrightarrow$ belief propagation (BP)

Question

- Why do polar codes achieve symmetric capacity with SC decoding ?
- Which bits should be chosen as information bits ?

$\Downarrow$

Channel polarization

Channel polarization

- U_0^{N-1}: Uniform random variable on $\{0,1\}^N$.
- $X_0^{N-1} := U_0^{N-1} G_N$.
- Y_0^{N-1}: Random variable corresponding output of W^n

$$NI(W) = I(U_0^{N-1}; Y_0^{N-1}) = \sum_{i=0}^{N-1} I(U_i; Y_0^{N-1}, U_0^{i-1}) = \sum_{i=0}^{N-1} I(W^{(i)})$$

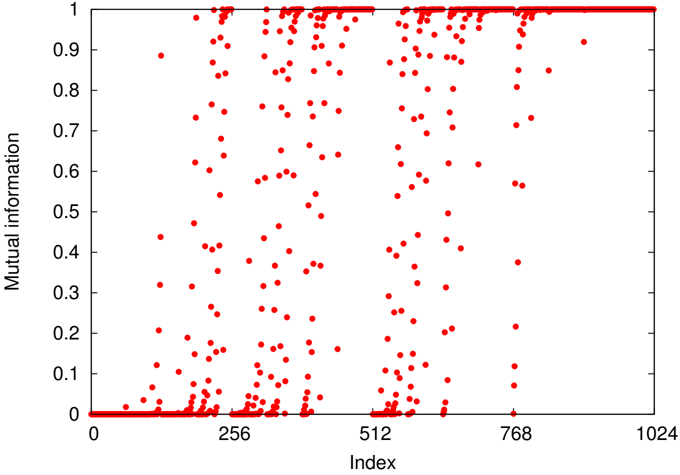
where $W^{(i)} : U_i \mapsto Y_0^{N-1}, U_0^{i-1}$.

Theorem (Channel polarization [Arıkan 2008])

$$\frac{\left|\left\{i \in \{0,1,\dots,N-1\} \mid I(W^{(i)}) > 1-\epsilon\right\}\right|}{N} = I(W)$$

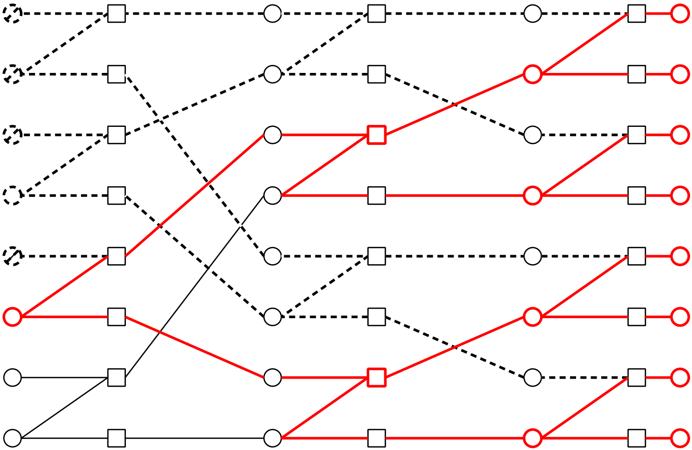
$$\frac{\left|\left\{i \in \{0,1,\dots,N-1\} \mid I(W^{(i)}) < \epsilon\right\}\right|}{N} = 1 - I(W)$$

Channel polarization



$$W = \mathrm{BEC}(0.4),\ I(W) = 0.6,\ N = 1024.$$

Process of subchannels



Binary erasure channel

$$P_\mathsf{e}(W^{(1)}) = \epsilon^2, \qquad P_\mathsf{e}(W^{(0)}) = 1-(1-\epsilon)^2$$

$$\frac{P_\mathsf{e}(W^{(0)}) + P_\mathsf{e}(W^{(1)})}{2} = \epsilon = P_\mathrm{e}(W)$$

Martingale convergence theorem

$B_1, \dots, B_n$: Uniform i.i.d. 0-1 random variables.

$Z_n := P_e(W^{(B_1)(B_2)\cdots(B_n)})$.

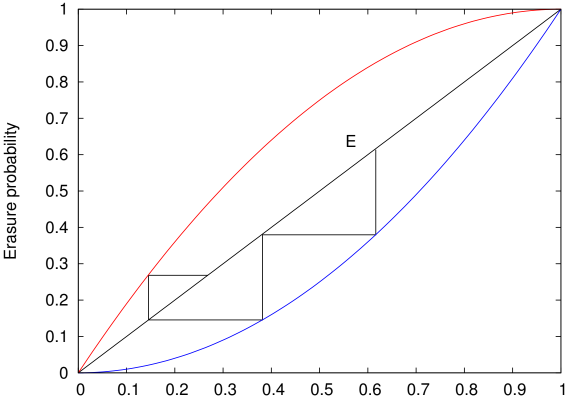
The random process Z_n is a martingale, i.e.,

$$\mathbb{E}[Z_{n+1} \mid B_1, \dots B_n] = \frac{Z_n^2 + 1 - (1 - Z_n)^2}{2} = Z_n$$

Theorem (Martingale convergence theorem)

If $\sup_n \mathbb{E}[X_n] < \infty$ then a martingale X_n converges almost surely.

Convergence to 0-1 random variable



$$Z_n = \begin{cases} Z_{n-1}^2, & \text{w.p.} \quad \frac{1}{2} \\ 1-(1-Z_{n-1})^2, & \text{w.p.} \quad \frac{1}{2} \end{cases}$$

$$Z_0 = \epsilon$$

Z_∞ takes 1 with prob. ϵ and takes 0 with prob. $1-\epsilon$.

From BEC to general channel

For general channel, $P_e(W^{(B_1)\cdots(B_n)})$ is not a martingale.

But, $I(W^{(B_1)\cdots(B_n)})$ is a martingale.

Theorem (Channel polarization [Arıkan 2008])

$$\frac{\left|\left\{i \in \{0,1,\ldots,N-1\} \mid I(W^{(i)}) > 1-\epsilon\right\}\right|}{N} = I(W)$$

$$\frac{\left|\left\{i \in \{0,1,\ldots,N-1\} \mid I(W^{(i)}) < \epsilon\right\}\right|}{N} = 1 - I(W).$$

Construction and error

Bits associated to high $I(W^{(i)})$ subchannels are chosen as information bits.

Bits associated to low $I(W^{(i)})$ subchannels are chosen as frozen bits.

$I(W^{(i)})$ (and also $P_e(W^{(i)})$) can be evaluated by density evolution [Mori and Tanaka 2009].

$$\begin{aligned}\mathcal{B}_i &:= \{\, u_1^N, y_1^N \mid \hat{u}_1^{i-1} = u_1^{i-1}, \hat{U}_i(y_1^N, \hat{u}_1^{i-1}) \neq u_i \,\}\\ &\subseteq \{\, u_1^N, y_1^N \mid \hat{U}_i(y_1^N, u_1^{i-1}) \neq u_i \,\} =: \mathcal{A}_i.\end{aligned}$$

$$P_{\mathrm{error}}(F) = \Pr\left(\bigcup_{i \in F^c} \mathcal{B}_i\right) = \sum_{i \in F^c} \Pr(\mathcal{B}_i) \leq \sum_{i \in F^c} \Pr(\mathcal{A}_i) = \sum_{i \in F^c} P_e(W_N^{(i)})$$

Table of contents

- Speed of polarization [Arıkan and Talatar 2008]
- $\ell \times \ell$ construction [Korada, Şaşoğlu, and Urbanke 2009]
- Detailed speed of polarization [Tanaka and Mori 2010], [Hassani and Urbanke 2010], [Hassani, Mori, Tanaka, and Urbanke 2012]
- Scaling of polarization [Koarada, Montanari, Telatar and Urbanke 2010]
- Compound capacity [Hassani, Korada and Urbanke 2009]
- Non-binary polar codes [Şaşoğlu, Telatar and Arıkan 2009], [Mori and Tanaka 2010]

Speed of polarization

[Arıkan and Telatar 2008] For any $\epsilon > 0$

$$\lim_{n\to\infty} \Pr\left(Z_n < 2^{-N^{\frac{1}{2}-\epsilon}}\right) = I(W)$$

$$\lim_{n\to\infty} \Pr\left(Z_n < 2^{-N^{\frac{1}{2}+\epsilon}}\right) = 0$$

Hence, error probability of SC decoding for polar codes is $o\left(2^{-N^{\frac{1}{2}-\epsilon}}\right)$ and $\omega\left(2^{-N^{\frac{1}{2}+\epsilon}}\right)$ for any $\epsilon > 0$.

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$\ell \times \ell$ matrix

Polar codes using an $\ell \times \ell$ matrix G instead of $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. [Korada, Şaşoğlu, and Urbanke 2009]

$E(G) \in (0, 1)$ is called an exponent of G if it holds

$$\lim_{n \to \infty} \Pr\left(Z_n < 2^{-N^{E(G)-\epsilon}} \right) = I(W)$$

$$\lim_{n \to \infty} \Pr\left(Z_n < 2^{-N^{E(G)+\epsilon}} \right) = 0$$

for any $\epsilon > 0$.

Obviously, $E\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right) = \frac{1}{2}$.

Exponent of matrix

[Korada, Şaşoğlu and Urbanke 2009]

$$E(G) = \frac{1}{\ell} \sum_{i=1}^{\ell} \log_\ell D_i$$

For $\ell \times \ell$ matrix G, partial distance D_i is defined as

$$\begin{aligned} D_i &:= d(g_i, \langle g_{i+1}, \ldots, g_\ell \rangle), \qquad i = 1, \ldots, \ell - 1 \\ D_\ell &:= d(g_\ell, 0) \end{aligned}$$

Here, g_i is i-th row of G and $\langle g_i, \ldots, g_\ell \rangle$ is the subcode spanned by $g_i, \ldots, g_\ell$.

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$D_1 = 1, D_2 = 1, D_3 = 3$ $\qquad$ $D_1 = 1, D_2 = 2, D_3 = 2$

Exponent of matrix

$$E(G) = \frac{1}{\ell} \sum_{i=1}^{\ell} \log_\ell D_i$$

For

$$E_\ell := \max_{G \in \{0,1\}^{\ell \times \ell}} E(G) \qquad \text{it holds} \qquad \lim_{\ell \to \infty} E_\ell = 1$$

$E_\ell \le \frac{1}{2}$ for $\ell \le 14$ and $E_{16} = 0.51828 > \frac{1}{2}$.

[Korada, Şaşoğlu, and Urbanke 2009]

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Detailed speed of polarization

[Tanaka and Mori 2010] [Hassani and Urbanke 2010]
[Hassani, Mori, Tanaka and Urbanke 2011]
For $R \in (0,1)$,

$$\lim_{n\to\infty} \Pr\left(Z(W_n) \le 2^{-\ell^{nE(G)+\sqrt{nV(G)}Q^{-1}(R/I(W))+f(n)}} \right) = R$$

for any $f(n) = o(\sqrt{n})$ where

$$Q(x) := \int_x^\infty \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2} \mathrm{d}x$$

$$E(G) := \frac{1}{\ell} \sum_{i=1}^{\ell} \log_\ell D_i(G)$$

$$V(G) := \frac{1}{\ell} \sum_{i=1}^{\ell} (\log_\ell D_i(G) - E(G))^2 .$$

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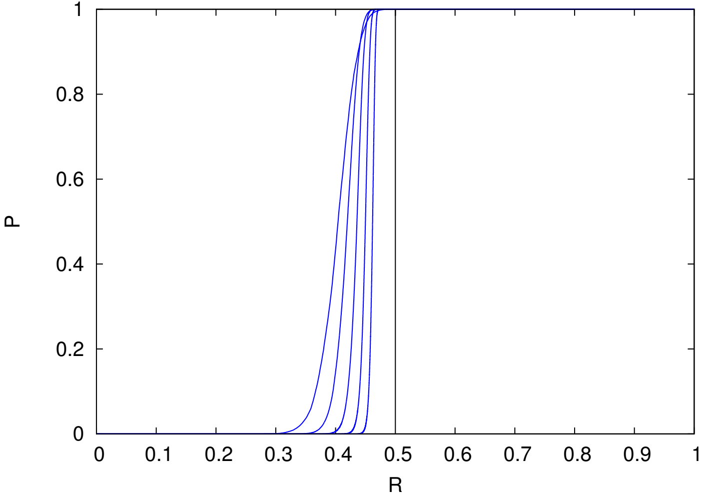
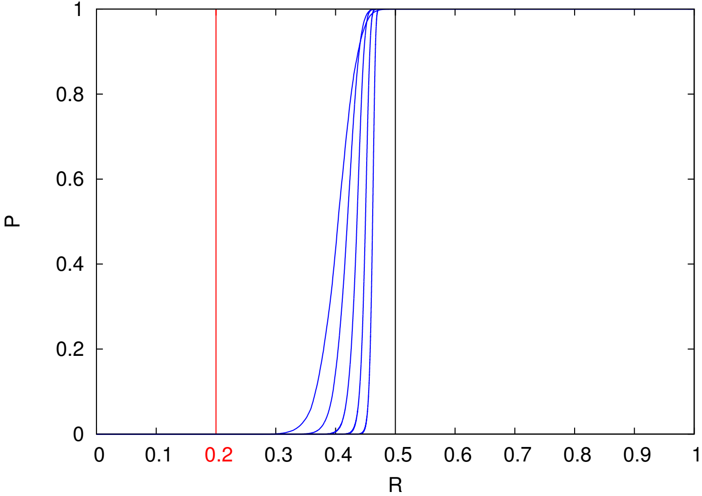
Scaling of polar codes BEC($\epsilon = 0.5$)



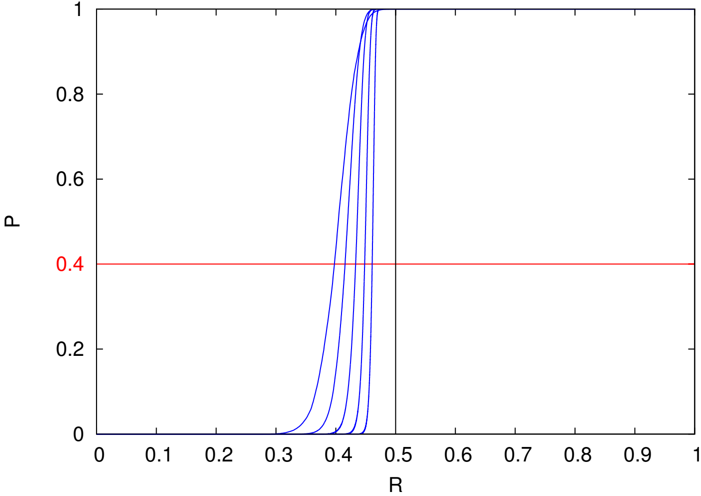
$N = 2^{10}, 2^{12}, 2^{14}, 2^{16}, 2^{18}$

Scaling of polar codes BEC($\epsilon = 0.5$)



$N \to \infty$ while R fixed $\iff$ Shannon–Gallager type analysis

Scaling of polar codes BEC($\epsilon = 0.5$)



$N \to \infty$ while P_e fixed $\Longleftrightarrow$ Weiss–Dobrushin–Strassen type analysis

Scaling of polarization

[Korada, Montanari, Telatar, and Urbanke 2010]
[Hassani, Alishahi, and Urbanke 2010]

For arbitrary fixed $a \in (0,1)$

$$F_N(\epsilon) := \Pr(\epsilon \leq Z(W_n) \leq a).$$

Scaling Assumption:
There exists $\mu > 0$ (called a scaling parameter) such that for any $\epsilon \in (0, a]$,

$$F(\epsilon) := \lim_{N\to\infty} N^{\frac{1}{\mu}} F_N(\epsilon) \in (0, \infty).$$

If the scaling assumption holds,

$$F^{-1}(N^{\frac{1}{\mu}}(I(W) - R)) \leq P_{\mathrm{e}}.$$

Scaling parameter of polar codes

[Korada, Montanari, Telatar, and Urbanke 2010]
[Hassani, Alishahi, and Urbanke 2010]

$$P_\mathrm{e} \geq F^{-1}(N^{\frac{1}{\mu}}(I(W) - R))$$
$$NR \leq NI(W) - N^{1-\frac{1}{\mu}} F(P_\mathrm{e})$$

This is Weiss–Dobrushin–Strassen type analysis.

From the scaling assumption

$$-\frac{1}{\mu} = \lim_{n\to\infty} \frac{1}{n} \log \Pr(a \leq Z(W_n) \leq b)$$

It can be evaluated for BEC, $1/\mu \approx 0.2757$.
By Gaussian approximation, for AWGN channel,$1/\mu \approx 0.2497$.
Random codes and LDPC codes have $1/\mu = 0.5$.

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- Non-binary polar codes [Şaşoğlu, Telatar and Arıkan 2009], [Mori and Tanaka 2010]

Compound capacity of polar codes

[Hassani, Korada, and Urbanke 2009]

$\mathcal{W}$: A set of channels

$$C(\mathcal{W}) = \max_{P_X} \inf_{W \in \mathcal{W}} I(X;Y)$$

The compound capacity of polar codes with SC decoding is

$$\lim_{N\to\infty} \sum_{i=1}^{N} \inf_{W \in \mathcal{W}} I(W_N^{(i)})$$

For BEC(0.5) and BSC(0.11002), the compound capacity is about 0.4816.

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Polarization by non-binary matrix

The matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

on the commutative ring $\mathbb{Z}/q\mathbb{Z}$ polarizes any channel if and only if q is a prime.
[Şaşoğlu, Telatar, and Arıkan 2009]

The matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & \gamma \end{bmatrix}$$

on the field $\mathbb{F}_q$ polarizes any channel if and only if $\mathbb{F}_p(\gamma) = \mathbb{F}_q$.
[Mori and Tanaka 2010]

Necessary and sufficient condition for $\ell \times \ell$ matrix on $\mathbb{F}_q$ is also obtained in [Mori and Tanaka 2012].

Reed-Solomon matrix [Mori and Tanaka 2010]

Let α be a primitive element of $\mathbb{F}_q$.
A Reed-Solomon matrix $G_{\mathrm{RS}}(q)$ is defined as

$$
\begin{array}{c|cccccc}
 & \alpha^{q-2} & \alpha^{q-3} & \cdots & \alpha & 1 & 0 \\
\hline
X^{q-1} & 1 & 1 & \cdots & 1 & 1 & 0 \\
X^{q-2} & \alpha^{(q-2)(q-2)} & \alpha^{(q-3)(q-2)} & \cdots & \alpha^{q-2} & 1 & 0 \\
X^{q-3} & \alpha^{(q-2)(q-3)} & \alpha^{(q-3)(q-3)} & \cdots & \alpha^{q-3} & 1 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
X & \alpha^{q-2} & \alpha^{q-3} & \cdots & \alpha & 1 & 0 \\
1 & 1 & 1 & \cdots & 1 & 1 & 1
\end{array}.
$$

Submatrix which consists of ith row to the last row is a generator matrix of extended Reed-Solomon code.
The size ℓ of RS matrix is q.
Since $G_{\mathrm{RS}}(2) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, RS matrix can be regarded as a generalization of Arıkan's binary matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Since $D_i = i + 1$, $E(G_{\mathrm{RS}}(q)) = \frac{\log(q!)}{q \log q}$

Exponent of Reed-Solomon matrix

$$E(G_{\mathrm{RS}}(q)) = \frac{\log(q!)}{q \log q}$$

q	2	4	16	64	256
$E(G_{\mathrm{RS}}(q))$	0.5	0.573120	0.691408	0.770821	0.822264

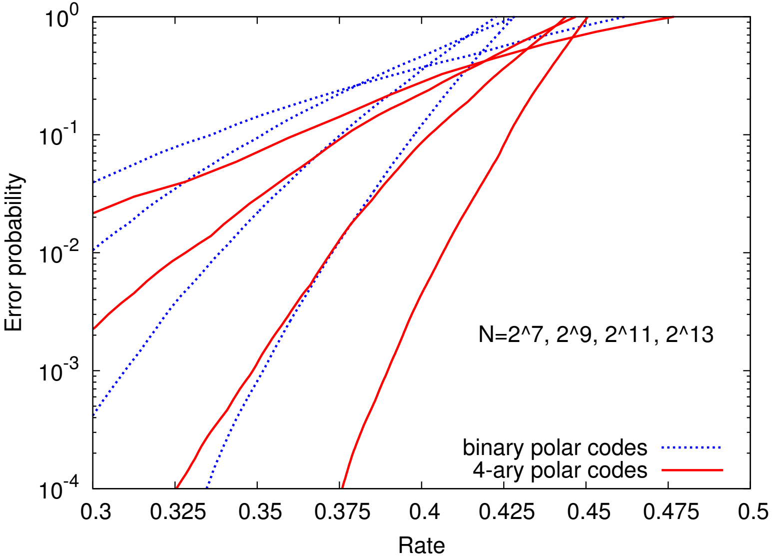
$$\lim_{q \to \infty} E(G_{\mathrm{RS}}(q)) = 1$$

The exponent of binary matrix of size smaller than 32 is smaller than 0.55
[Korada, Şaşoğlu, and Urbanke 2009]

Reed-Solomon matrix is useful for obtaining large exponent !

How about the performance for finite blocklength ?

Simulation result on BAWGNC ($I(W) = 0.5$)



Polar codes and Reed-Muller codes: binary case

[Arıkan 2009]

$$X: \; 1 \quad 0 \qquad\qquad (X_2, X_1): (1,1)(1,0)(0,1)(0,0)$$

$$\begin{array}{cc} X & \\ 1 & \end{array} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \begin{array}{r} X_2X_1 \\ X_2 \\ X_1 \\ 1 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{array}{l} 00 \\ 01 \\ 10 \\ 11 \end{array}$$

Polar rule: $\{i \in \{0, \ldots, 2^n - 1\} \mid P_e(W^{(i_1)\cdots(i_n)}) < \epsilon\}$

Reed-Muller rule: $\{i \in \{0, \ldots, 2^n - 1\} \mid i_1 + \cdots + i_n > k\}$

Binary polar codes using $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and binary Reed-Muller codes are similar.

Reed-Muller rule maximizes the minimum distance.

Polar codes using RS matrix and Reed-Muller codes: q-ary case

$$
\begin{array}{c|ccccccccc|c}
(X_2, X_1): & (2,2) & (2,1) & (2,0) & (1,2) & (1,1) & (1,0) & (0,2) & (0,1) & (0,0) & \\
\hline
X_2^2 X_1^2 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 00 \\
X_2^2 X_1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 01 \\
X_2^2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 02 \\
X_2 X_1^2 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 10 \\
X_2 X_1 & 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 11 \\
X_2 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 12 \\
X_1^2 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 20 \\
X_1 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 21 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 22
\end{array}
$$

Polar rule: $\{i \in \{0, \dots, q^n - 1\} \mid P_e(W^{(i_1)\cdots(i_n)}) < \epsilon\}$

Reed-Muller rule: $\{i \in \{0, \dots, q^n - 1\} \mid i_1 + \cdots + i_n > k\}$

Q-ary polar codes using $G_{\mathrm{RS}}(q)$ and q-ary Reed-Muller codes are also similar.

Hyperbolic rule: $\{i \in \{0, \dots, q^n - 1\} \mid (i_1 + 1) \cdots (i_n + 1) > k\}$

Hyperbolic rule maximizes the minimum distance

(Massey–Costello–Justesen codes, hyperbolic cascaded RS codes).

Summary

- Polar code is provably capacity-achieving codes with efficient decoding algorithm.
- Error probability of polar codes decays slowly if rate is close to the capacity.
- Asymptotic performance of polar codes can be improved by $\ell \times \ell$ matrix and non-binary matrix.
- Polar code is suitable for many problems, e.g., lossless and lossy source coding, problems with side information, multiple access channel, etc.